

Coverings of internal groupoids and crossed modules in the category of groups with operations

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Abstract

In this paper we prove some results on the covering morphisms of internal groupoids. We also give a result on the coverings of the crossed modules of groups with operations.

Key Words: Internal category, covering groupoid, group with operations, crossed module

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1 Introduction

A *groupoid* is a small category in which each morphism is an isomorphism [4, 18]. A *group-groupoid* is a group object in the category of groupoids [8]; equivalently, as is well known, it is an internal groupoid in the category of groups and, according to [25], it is an internal category in the category of groups. An alternative name, quite generally used, is “2-group”, see for example [2]. Recently the notion of monodromy for topological group-groupoids was introduced and investigated in [21], and normality and quotients in group-groupoids were studied in [22].

A *crossed module* defined by Whitehead in [26, 27] can be viewed as a 2-dimensional group [3] and has been widely used in: homotopy theory [5]; the theory of identities among relations for group presentations [6]; algebraic K -theory [16]; and homological algebra, [15, 17]. See [5, p. 49] for a discussion of the relation of crossed modules to crossed squares and so to homotopy 3-types.

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In [8, Theorem 1] Brown and Spencer proved that the category of internal categories within the groups, i.e. group-groupoids, is equivalent to the category of crossed modules of groups. Then in [25, Section 3], Porter proved that a similar result holds for certain algebraic categories \mathcal{C} , introduced by Orzech [23], the definition of which was adapted by him and called category of groups with operations. Applying Porter's result, the study of internal category theory in \mathcal{C} was continued in the works of Datuashvili [10] and [12]. Moreover, she developed the cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [9, 11]. In a similar way, the results of [8] and [25] enabled us to prove some properties of covering groupoids for internal groupoids.

On the other hand, there are some important results on the covering groupoids of group-groupoids. One is that the group structure of a group-groupoid lifts to its some covering groupoids, i.e. if G is a group-groupoid, 0 is the identity element of the additive group G_0 of objects and $p: (\tilde{G}, \tilde{0}) \rightarrow (G, 0)$ is a covering morphism of groupoids whose characteristic group is a subgroup of G , then \tilde{G} becomes a group-groupoid such that p is a morphism of group-groupoids [1, Theorem 2.7]. Another is that if X is a topological group whose topology is semi-locally simply connected, then the category $\text{Cov}_{\text{TopGr}}/X$ of covers of X in the category of topological groups and the category $\text{Cov}_{\text{GrGd}}/\pi(X)$ of covers of $\pi(X)$ in the category of group-groupoids are equivalent [7, Proposition 2.3]. The another one is that for a group-groupoid G the category $\text{Cov}_{\text{GrGd}}/G$ of covers of G in the category of group-groupoids is equivalent to the category $\text{Act}_{\text{GrGd}}/G$ of group-groupoid actions of G on groups [7, Proposition 3.1]. The final one is that if X is a topological group, then the restriction $d_1: St_{\pi(X)}e \rightarrow X$ of the final point map to the star at the identity $e \in X$ is a crossed module of groups and the category $\text{Cov}_{\text{GrGd}}/\pi(X)$ of covers of $\pi(X)$ in the category of group-groupoids and the category of covers of $St_{\pi(X)}e \rightarrow X$ in the category of crossed modules within groups are equivalent [7, Corollary 4.3].

The object of this paper is to prove that the above-mentioned results can be generalized to a wide class of algebraic categories, which include categories of groups, rings, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras, alternative algebras and others. These are conveniently handled by working in a category \mathcal{C} . For the first result we prove that if G is an internal groupoid in \mathcal{C} and $p: (\tilde{G}, \tilde{0}) \rightarrow (G, 0)$ is a covering morphism of groupoids whose characteristic group is closed under the group operations of G , then \tilde{G} becomes an internal groupoid such that p is a morphism of internal groupoids. For the second result we prove that if X is a topological group with operations whose topology is semi-locally simply connected, then the category $\text{Cov}_{\text{TopGr}}/X$ of covers of X in the category of topological groups with operations and the category $\text{Cov}_{\text{Cat}(\mathcal{C})}/\pi(X)$ of covers of $\pi(X)$ in the category $\text{Cat}(\mathcal{C})$ of internal categories, equivalently, internal groupoids in \mathcal{C} (see the

note after Definition 3.2), are equivalent. For the third one we prove that if G is an internal groupoid in \mathbf{C} , then the category $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ of covers of G in the category of internal groupoids in \mathbf{C} is equivalent to the category $\text{Act}_{\text{Cat}(\mathbf{C})}/G$ of internal groupoid actions of G on groups with operations. Finally we prove that if G is an internal groupoid in \mathbf{C} and $\alpha: A \rightarrow B$ is the crossed module corresponding to G , then the category $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ and the category $\text{Cov}_{\text{XMod}}/(\alpha: A \rightarrow B)$ of covering crossed modules of $\alpha: A \rightarrow B$ are equivalent.

2 Covering morphisms of groupoids

As is defined in [4, 18], a groupoid G has a set G of morphisms, which we call just *elements* of G , a set G_0 of *objects* together with maps $d_0, d_1: G \rightarrow G_0$ and $\epsilon: G_0 \rightarrow G$ such that $d_0\epsilon = d_1\epsilon = 1_{G_0}$. The maps d_0, d_1 are called *initial* and *final* point maps, respectively, and the map ϵ is called the *object inclusion*. If $a, b \in G$ and $d_1(a) = d_0(b)$, then the *composite* $a \circ b$ exists such that $d_0(a \circ b) = d_0(a)$ and $d_1(a \circ b) = d_1(b)$. So there exists a partial composition defined by $G_{d_1} \times_{d_0} G \rightarrow G, (a, b) \mapsto a \circ b$, where $G_{d_1} \times_{d_0} G$ is the pullback of d_1 and d_0 . Further, this partial composition is associative, for $x \in G_0$ the element $\epsilon(x)$ acts as the identity, and each element a has an inverse a^{-1} such that $d_0(a^{-1}) = d_1(a)$, $d_1(a^{-1}) = d_0(a)$, $a \circ a^{-1} = \epsilon d_0(a)$ and $a^{-1} \circ a = \epsilon d_1(a)$. The map $G \rightarrow G, a \mapsto a^{-1}$ is called the *inversion*.

In a groupoid G , for $x, y \in G_0$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y . According to [4] G is *transitive* if for all $x, y \in G_0$, the set $G(x, y)$ is not empty; for $x \in G_0$ the *star* of x is defined as $\{a \in G \mid s(a) = x\}$ and denoted as $St_G x$; and the *object group* at x is defined as $G(x, x)$ and denoted as $G(x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f: H \rightarrow G$ and $f_0: H_0 \rightarrow G_0$ such that $d_0 f = f_0 d_0$, $d_1 f = f_0 d_1$, $f\epsilon = \epsilon f_0$ and $f(a \circ b) = f(a) \circ f(b)$ for all $(a, b) \in H_{d_1} \times_{d_0} H$. For such a morphism we simply write $f: H \rightarrow G$.

Let $p: \tilde{G} \rightarrow G$ be a morphism of groupoids. Then p is called a *covering morphism* and \tilde{G} a *covering groupoid* of G if for each $\tilde{x} \in \tilde{G}_0$ the restriction $St_{\tilde{G}} \tilde{x} \rightarrow St_G p(\tilde{x})$ is bijective. A covering morphism $p: \tilde{G} \rightarrow G$ is called *transitive* if both \tilde{G} and G are transitive. A transitive covering morphism $p: \tilde{G} \rightarrow G$ is called *universal* if \tilde{G} covers every cover of G , i.e. if for every covering morphism $q: \tilde{H} \rightarrow G$ there is a unique morphism of groupoids $\tilde{p}: \tilde{G} \rightarrow \tilde{H}$ such that $q\tilde{p} = p$ (and hence \tilde{p} is also a covering morphism), this is equivalent to that for $\tilde{x}, \tilde{y} \in O_{\tilde{G}}$ the set $\tilde{G}(\tilde{x}, \tilde{y})$ has not more than one element.

A morphism $p: (\tilde{G}, \tilde{x}) \rightarrow (G, x)$ of pointed groupoids is called a *covering morphism* if the morphism $p: \tilde{G} \rightarrow G$ is a covering morphism. Let $p: (\tilde{G}, \tilde{x}) \rightarrow (G, x)$ be a covering morphism of groupoids and $f: (H, z) \rightarrow (G, x)$ a morphism of groupoids. We say f lifts to p if there exists a unique morphism $\tilde{f}: (H, z) \rightarrow (\tilde{G}, \tilde{x})$ such that $f = p\tilde{f}$. For any groupoid morphism

$p: \tilde{G} \rightarrow G$ and an object \tilde{x} of \tilde{G} we call the subgroup $p(\tilde{G}(\tilde{x}))$ of $G(p\tilde{x})$ the *characteristic group* of p at \tilde{x} .

The following result gives a criterion on the liftings of morphisms [4, 10.3.3].

Theorem 2.1. *Let $p: (\tilde{G}, \tilde{x}) \rightarrow (G, x)$ be a covering morphism of groupoids and $f: (H, z) \rightarrow (G, x)$ a morphism such that H is transitive. Then the morphism $f: (H, z) \rightarrow (G, x)$ lifts to a morphism $\tilde{f}: (H, z) \rightarrow (\tilde{G}, \tilde{x})$ if and only if the characteristic group of f is contained in that of p ; and if this lifting exists, then it is unique.*

As a result of Theorem 2.1 the following corollary is stated in [4, 10.3.4].

Corollary 2.1. *Let $p: (\tilde{G}, \tilde{x}) \rightarrow (G, x)$ and $q: (\tilde{H}, \tilde{z}) \rightarrow (G, x)$ be transitive covering morphisms with characteristic groups C and D , respectively. If $C \subseteq D$, then there is a unique covering morphism $r: (\tilde{G}, \tilde{x}) \rightarrow (\tilde{H}, \tilde{z})$ such that $p = qr$. If $C = D$, then r is an isomorphism.*

The action of a groupoid on a set is defined in [4, p.373] as follows.

Definition 2.1. Let G be a groupoid. An *action* of G on a set consists of a set X , a function $\theta: X \rightarrow G_0$ and a function $\varphi: X_\theta \times_{d_0} G \rightarrow X$, $(x, a) \mapsto xa$ defined on the pullback $X_\theta \times_{d_0} G$ of θ and d_0 such that

1. $\theta(xa) = d_1(a)$ for $(x, a) \in X_\theta \times_{d_0} G$;
2. $x(a \circ b) = (xa)b$ for $(a, b) \in G_{d_1} \times_{d_0} G$ and $(x, a) \in X_\theta \times_{d_0} G$;
3. $x\epsilon(\theta(x)) = x$ for $x \in X$.

According to [4], given such an action, the *action groupoid* $G \ltimes X$ is defined to be the groupoid with object set X and with elements of $(G \ltimes X)(x, y)$ the pairs (a, x) such that $a \in G(\theta(x), \theta(y))$ and $xa = y$. The groupoid composite is defined to be

$$(a, x) \circ (b, y) = (a \circ b, x)$$

when $y = xa$.

The following result is from [4, 10.4.3]. We need some details of its proof in the proofs of Theorem 3.3 and Theorem 3.5.

Theorem 2.2. *Let x be an object of a transitive groupoid G , and let C be a subgroup of the object group $G(x)$. Then there exists a covering morphism $q: (\tilde{G}_C, \tilde{x}) \rightarrow (G, x)$ with characteristic group C .*

Proof. We give a sketch proof for a technical method: Let X be the set of cosets $C \circ a = \{c \circ a \mid c \in C\}$ for a in $St_G x$. Let $\theta: X \rightarrow G_0$ be a map, which sends $C \circ a$ to the final point of a . The function θ is well defined because if $C \circ a = C \circ b$ then $t(a) = t(b)$. The groupoid G acts on X by

$$\varphi: X_\theta \times_{d_0} G \rightarrow X, (C \circ a, g) \mapsto C \circ (a \circ g).$$

The required groupoid \tilde{G}_C is taken to be the action groupoid $G \ltimes X$. Then the projection $q: \tilde{G}_C \rightarrow G$ given on objects by $\theta: X \rightarrow G_0$ and on elements by $(g, C \circ a) \mapsto g$, is a covering morphism of groupoids and has the characteristic group C . Here the groupoid composite on \tilde{G}_C is defined by

$$(g, C \circ a) \circ (h, C \circ b) = (g \circ h, C \circ a)$$

whenever $C \circ b = C \circ a \circ g$. The required object $\tilde{x} \in \tilde{G}_C$ is the coset C . □

3 Groups with operations and internal categories

The idea of the definition of categories of groups with operations comes from Higgins [14] and Orzech [23]; and the definition below is from Porter [25] and Datuashvili [13, p. 21], which is adapted from Orzech [23].

From now on \mathbf{C} will be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold: If Ω_i is the set of i -ary operations in Ω , then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively 0 , $-$ and $+$ are the elements of Ω_0 , Ω_1 and Ω_2 respectively. Let $\Omega'_2 = \Omega_2 \setminus \{+\}$, $\Omega'_1 = \Omega_1 \setminus \{-\}$ and assume that if $\star \in \Omega'_2$, then \star° defined by $a \star^\circ b = b \star a$ is also in Ω'_2 . Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega'_2$, E includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega'_1$ and $\star \in \Omega'_2$, E includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

A category satisfying the conditions (a)-(d) is called a *category of groups with operations*.

A *morphism* between any two objects of \mathbf{C} is a group homomorphism, which preserves the operations from Ω'_1 and Ω'_2 .

Remark 3.1. The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.1. The categories of groups, rings generally without identity, R -modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

If A and B are objects of \mathcal{C} an *extension* of A by B is an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{p} B \longrightarrow 0$$

in which p is surjective and ι is the kernel of p . It is *split* if there is a morphism $s: B \rightarrow E$ such that $ps = \text{id}_B$. A split extension of B by A is called a B -*structure* on A . Given such a B -structure on A , we get the actions of B on A corresponding to the operations in \mathcal{C} . For any $b \in B$, $a \in A$ and $\star \in \Omega'_2$ we have the actions called *derived actions* by Orzech [23, p. 293]

$$\begin{aligned} b \cdot a &= s(b) + a - s(b), \\ b \star a &= s(b) \star a. \end{aligned}$$

Theorem 3.1 ([23, Theorem 2.4]). *A set of actions (one for each operation in Ω_2) is a set of derived actions if and only if the semidirect product $B \ltimes A$ with underlying set $B \times A$ and operations*

$$\begin{aligned} (b', a') + (b, a) &= (b' + b, a' \cdot b + a), \\ (b', a') \star (b, a) &= (b' \star b, b' \star a + a' \star b + a' \star a) \end{aligned}$$

is an object in \mathcal{C} .

Definition 3.1 ([23, Definition 1.5]). Let X be an object of \mathcal{C} . A subobject A of X is called an *ideal* if it is the kernel of some morphism.

The concept of ideal is characterized as follows.

Theorem 3.2 ([23, Theorem 1.7]). *Let A be a subobject of the object X . Then A is an ideal of X if and only if the following conditions hold:*

1. *A is a normal subgroup of X ;*
2. *$a \star x \in A$ for $a \in A$, $x \in X$ and $\star \in \Omega'_2$.*

We define the category of topological groups with operations as follows.

In the rest of the paper \mathcal{TC} will denote the category of topological groups with a set Ω of continuous operations and with a set E of identities such that E includes the group laws such that the conditions (a)-(d) of Section 2 are satisfied.

Such a category is called a *category of topological groups with operations*.

A *morphism* between any two objects of \mathcal{TC} is a continuous group homomorphism, which preserves the operations in Ω'_1 and Ω'_2 .

The categories of topological groups, topological rings and topological R -modules are examples of categories of topological groups with operations.

The internal category in \mathbf{C} is defined in [25] as follows. We follow the notations of Section 1 for groupoids.

Definition 3.2. An *internal category* C in \mathbf{C} is a category in which the initial and final point maps $d_0, d_1: C \rightrightarrows C_0$, the object inclusion map $\epsilon: C_0 \rightarrow C$ and the partial composition $\circ: C_{d_1} \times_{d_0} C \rightarrow C, (a, b) \mapsto a \circ b$ are the morphisms in the category \mathbf{C} .

Note that since ϵ is a morphism in \mathbf{C} , $\epsilon(0) = 0$ and that the operation \circ being a morphism implies that for all $a, b, c, d \in C$ and $\star \in \Omega_2$,

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d) \quad (1)$$

whenever one side makes sense. This is called the *interchange law* [25].

We also note from [25] that any internal category in \mathbf{C} is an internal groupoid since, given $a \in C$, $a^{-1} = \epsilon d_1(a) - a + \epsilon d_0(a)$ satisfies $a^{-1} \circ a = \epsilon d_1(a)$ and $a \circ a^{-1} = \epsilon d_0(a)$. So we use the term *internal groupoid* rather than internal category and write G for an internal groupoid. For the category of internal groupoids in \mathbf{C} we use the same notation $\text{Cat}(\mathbf{C})$ as in [25]. Here a *morphism* $f: H \rightarrow G$ in $\text{Cat}(\mathbf{C})$ is the morphism of underlying groupoids and a morphism in \mathbf{C} .

In particular if \mathbf{C} is the category of groups, then an internal groupoid G in \mathbf{C} becomes a group-groupoid and in the case where \mathbf{C} is the category of rings, an internal groupoid in \mathbf{C} is a ring object in the category of groupoids [20].

Remark 3.2. We emphasize the following points from Definition 3.2:

- (i) By Definition 3.2 we know that in an internal groupoid G in \mathbf{C} , the initial and final point maps d_0 and d_1 , the object inclusion map ϵ are the morphisms in \mathbf{C} and the interchange law (1) is satisfied. Therefore in an internal groupoid G , the unary operations are endomorphisms of the underlying groupoid of G and the binary operations are morphisms from the underlying groupoid of $G \times G$ to the one of G .
- (ii) Let G be an internal groupoid in \mathbf{C} and $0 \in G_0$ the identity element. Then $\text{Ker} d_0 = \text{St}_G 0$, called in [4] the *transitivity component* or *connected component* of 0 , is also an internal groupoid which is also an ideal of G .

The following example plays the key rule in the proofs of Corollary 3.1 and Theorem 4.1.

Example 3.2. If X is an object of TC , then the fundamental groupoid $\pi(X)$ is an internal groupoid. Here the operations on $\pi(X)$ are induced by those of X . The details are straightforward.

Definition 3.3. A morphism $f: H \rightarrow G$ of internal groupoids in \mathcal{C} is called a *cover* (resp. *universal cover*) if it is a covering morphism (resp. universal covering morphism) on the underlying groupoids.

Since by Remark 3.2 (ii) for an internal groupoid G in \mathcal{C} , the star $St_G 0$ is also an internal groupoid, we have that if $f: H \rightarrow G$ is a covering morphism of internal groupoids, then the restriction of f to the stars $St_H 0 \rightarrow St_G 0$ is an isomorphism in \mathcal{C} .

A morphism $p: \tilde{X} \rightarrow X$ in \mathbf{TC} is called a *covering morphism* of topological groups with operations if it is a covering map on the underlying space.

Example 3.3. If $p: \tilde{X} \rightarrow X$ is a covering morphism of topological groups with operations, then the induced morphism $\pi(p): \pi(\tilde{X}) \rightarrow \pi(X)$ is a covering morphism of internal groupoids in \mathcal{C} .

In [7, 19] an action of a group-groupoid G on a group X is defined. We now define an action of an internal groupoid on a group with operations as follows.

Definition 3.4. Let G be an internal groupoid in \mathcal{C} and X an object of \mathcal{C} . If the underlying groupoid of G acts on the underlying set of X in the sense of Definition 2.1 so that the maps $\theta: X \rightarrow G_0$ and $\varphi: X_\theta \times_{d_0} G \rightarrow X, (x, a) \mapsto xa$ in the groupoid action are morphisms in \mathcal{C} , then we say that the internal groupoid G *acts* on the group with operations X via θ .

We write (X, θ, φ) for an action. Here note that $\varphi: X_\theta \times_{d_0} G \rightarrow X, (x, a) \mapsto xa$ is a morphism in \mathcal{C} if and only if

$$(x \star y)(a \star b) = (xa) \star (yb) \quad (2)$$

for $x, y \in X; a, b \in G$ and $\star \in \Omega_2$ whenever one side is defined.

Example 3.4. Let G and \tilde{G} be internal groupoids in \mathcal{C} and let $p: \tilde{G} \rightarrow G$ be a covering morphism of internal groupoids. Then the internal groupoid G acts on the group with operations $X = \tilde{G}_0$ via $p_0: X \rightarrow G_0$ assigning to $x \in X$ and $a \in St_G p(x)$ the target of the unique lifting \tilde{a} in \tilde{G} of a with source x . Clearly, the underlying groupoid of G acts on the underlying set and by evaluating the uniqueness of the lifting, condition (2) is satisfied for $x, y \in X$ and $a, b \in G$ whenever one side is defined.

We use Theorem 2.2 to prove the following result for internal groupoids.

Theorem 3.3. Let G be an internal groupoid in \mathcal{C} with transitive underlying groupoid, and $0 \in G_0$ the identity element of the additive operation. Let $G(0)$ be the object group at $0 \in G_0$, which is a group with operations and C a subobject of $G(0)$. Suppose that X is the set of cosets $C \circ a = \{c \circ a \mid c \in C\}$

for a in $St_G 0$. Then X becomes a group with operations such that the internal groupoid G acts on X as a group with operations.

Proof. Define 2-ary operations on X by

$$(C \circ a) \tilde{\star} (C \circ b) = C \circ (a \star b)$$

for $C \circ a, C \circ b \in X$. We now prove that these operations are well defined. If $C \circ a, C \circ b \in X$, then $a, b \in St_G 0$, $a \star b \in St_G 0$ and so $C \circ (a \star b) \in X$. Further if $C \circ a = C \circ a'$ and $C \circ b = C \circ b'$, then $a' \circ a^{-1}, b' \circ b^{-1} \in C$ and by the interchange law (1) in G

$$(a' \star b') \circ (a \star b)^{-1} = (a' \star b') \circ (a^{-1} \star b^{-1}) = (a' \circ a^{-1}) \star (b' \circ b^{-1}).$$

Since C is a subobject, we have that $(a' \star b') \circ (a \star b)^{-1} \in C$ and therefore $C \circ (a \star b) = C \circ (a' \star b')$.

Define 1-ary operations on X by

$$\tilde{\omega}(C \circ a) = C \circ \omega(a).$$

If $C \circ a = C \circ b$, then $b \circ a^{-1} \in C$ and

$$\omega(b \circ a^{-1}) = \omega(b) \circ \omega(a^{-1}) = \omega(b) \circ (\omega a)^{-1} \in C.$$

So $C \circ \omega(a) = C \circ \omega(b)$ and hence these operations are well defined. Since G is a group with operations it has a set E of identities including group axioms. So, the same type of identities including group axioms and the other axioms (a)-(d) of Section 2 are satisfied for X . Therefore X becomes a group with operations. We know from the proof of Theorem 2.2 that the underlying groupoid G acts on the set X . In addition to these, it is straightforward to show that by the interchange law (1) in G , the condition

$$((C \circ a) \tilde{\star} (C \circ b))(g \star h) = ((C \circ a) \circ g) \tilde{\star} ((C \circ b) \circ h)$$

is satisfied for $g, h \in G$ whenever the right side is defined. □

Theorem 3.4. *Let G be an internal category in \mathcal{C} and X an object of \mathcal{C} . Suppose that the internal groupoid G acts on the group with operations X . Then the action groupoid $G \ltimes X$ defined in Section 1 becomes an internal groupoid in \mathcal{C} such that the projection $p: G \ltimes X \rightarrow G$ is a morphism of internal groupoids.*

Proof. We first prove that the action groupoid $\tilde{G} = G \ltimes X$ is a group with operations in \mathcal{C} . For this, 1-ary operations are defined by $\tilde{\omega}(a, x) = (\omega(a), \omega(x))$ for Ω_1 and 2-ary operations

are defined by $(a, x)\tilde{\star}(b, y) = (a \star b, x \star y)$ for $\star \in \Omega_2$. Then the axioms (a)-(d) of Section 2 for these operations defined on \tilde{G} are satisfied. Since G and X are objects of \mathcal{C} , they have the same type of identities including group axioms. So, the same type of identities including group axioms are satisfied for \tilde{G} .

The initial and final point maps $d_0, d_1: \tilde{G} \rightrightarrows X$, the object inclusion map $\epsilon: X \rightarrow \tilde{G}$ and the partial composition $\circ: \tilde{G}_{d_1} \times_{d_0} \tilde{G} \rightarrow \tilde{G}$, $(\tilde{a}, \tilde{b}) \mapsto \tilde{a} \circ \tilde{b}$ are the morphisms in the category \mathcal{C} because the same maps for G are morphisms in \mathcal{C} .

By the interchange law (1) in G , the partial composition $\circ: \tilde{G}_{d_1} \times_{d_0} \tilde{G} \rightarrow \tilde{G}$, $(\tilde{a}, \tilde{b}) \mapsto \tilde{a} \circ \tilde{b}$ is a morphism in \mathcal{C} and, by condition (2) in G , the final point map $d_0: \tilde{G} \rightarrow X$ becomes a morphism in \mathcal{C} . The rest of the proof is straightforward. \square

Definition 3.5. Let \tilde{G} be a groupoid, G an internal groupoid in \mathcal{C} and $0 \in G_0$ the identity of the additive operation. Suppose that $p: (\tilde{G}, \tilde{0}) \rightarrow (G, 0)$ is a covering morphism of groupoids. We say that the internal groupoid structure of G *lifts* to \tilde{G} if \tilde{G} is an internal groupoid in \mathcal{C} such that p is a morphism of internal groupoids in \mathcal{C} .

We use Theorems 3.3 and 3.4 to prove that the internal groupoid structure of an internal category G lifts to a covering groupoid.

Theorem 3.5. *Let \tilde{G} be a groupoid and G an internal groupoid in \mathcal{C} whose underlying groupoid is transitive. Suppose that $p: (\tilde{G}, \tilde{0}) \rightarrow (G, 0)$ is a covering morphism of underlying groupoids such that the characteristic group C of p is a subobject of $G(0)$. Then the internal groupoid structure of G lifts to \tilde{G} with identity $\tilde{0}$.*

Proof. Let C be the characteristic group of $p: (\tilde{G}, \tilde{0}) \rightarrow (G, 0)$. Then by Theorem 2.2 we have a covering morphism of groupoids $q: (\tilde{G}_C, \tilde{x}) \rightarrow (G, 0)$ which has the characteristic group C . So by Corollary 2.1 the covering morphisms p and q are isomorphic. Therefore it is sufficient to prove that the internal groupoid structure of G lifts to $\tilde{G}_C = G \ltimes X$. By Theorem 3.3, X is a group with operations and the internal groupoid G acts on X . By Theorem 3.4, the internal groupoid structure of G lifts to $\tilde{G}_C = G \ltimes X$. \square

From Theorem 3.5 we obtain the following corollary.

Corollary 3.1. *Let X be an object of \mathcal{TC} whose underlying space is connected. Suppose that \tilde{X} is a simply connected topological space and $p: \tilde{X} \rightarrow X$ is a covering map from \tilde{X} to the underlying topology of X . Let 0 be the identity element of the additive group of X and $\tilde{0} \in \tilde{X}$ such that $p(\tilde{0}) = 0$. Then \tilde{X} becomes a topological group with operations such that $\tilde{0}$ is the identity element of the group structure of \tilde{X} and p is a morphism of topological groups with operations.*

Proof. Since $p: \tilde{X} \rightarrow X$ is a covering map, the induced morphism $\pi(p): \pi(\tilde{X}) \rightarrow \pi(X)$ becomes a covering morphism of groupoids with a trivial characteristic group. Since X is a topological group with operations, by Example 3.2 $\pi(X)$ is an internal groupoid and since X is path connected, the groupoid $\pi(X)$ is transitive. So by Theorem 3.5, the internal groupoid structure of $\pi(X)$ lifts to $\pi(\tilde{X})$. So, we have the structure of a group with operations on \tilde{X} . By [4, 10.5.5] the topology on \tilde{X} is the *lifted topology* obtained by the covering morphism $\pi(p): \pi(\tilde{X}) \rightarrow \pi(X)$ and the group operations are continuous with this topology. So, \tilde{X} becomes a topological group with operations. \square

4 The equivalence of the categories

Let X be an object of TC. So, by Example 3.2, $\pi(X)$ is an internal groupoid. Then we have a category Cov_{TC}/X of covers of X in the category TC of topological groups with operations and a category $\text{Cov}_{\text{Cat}(\mathbb{C})}/\pi(X)$ of covers of $\pi(X)$ in the category $\text{Cat}(\mathbb{C})$ of internal groupoids in \mathbb{C} .

Theorem 4.1. *Let X be an object of TC such that the underlying topology of X has a simply connected cover. Then the categories Cov_{TC}/X and $\text{Cov}_{\text{Cat}(\mathbb{C})}/\pi(X)$ are equivalent.*

Proof. Define a functor

$$\pi: \text{Cov}_{\text{TC}}/X \rightarrow \text{Cov}_{\text{Cat}(\mathbb{C})}/\pi(X)$$

as follows: suppose that $p: \tilde{X} \rightarrow X$ is a covering morphism of topological groups with operations. Then by Example 3.3, the induced morphism $\pi(p): \pi(\tilde{X}) \rightarrow \pi(X)$ is a morphism of internal groupoids and a covering morphism on the underlying groupoids which preserves the group operations. Therefore $\pi(p): \pi(\tilde{X}) \rightarrow \pi(X)$ becomes a covering morphism of internal groupoids.

We now define another functor

$$\eta: \text{Cov}_{\text{Cat}(\mathbb{C})}/\pi(X) \rightarrow \text{Cov}_{\text{TC}}/X$$

as follows: suppose that $q: \tilde{G} \rightarrow \pi(X)$ is a covering morphism of internal groupoids. By the lifted topology [4, 10.5.5] on $\tilde{X} = \tilde{G}_0$ there is an isomorphism $\alpha: \tilde{G} \rightarrow \pi(\tilde{X})$ of groupoids such that $p = O_q: \tilde{X} \rightarrow X$ is a covering map and $q = \pi(p) \alpha$. Hence the group operations on \tilde{G} transport via α to $\pi(\tilde{X})$ such that $\pi(\tilde{X})$ is an internal groupoid. So we have 2-ary operations

$$\tilde{\star}: \pi(\tilde{X}) \times \pi(\tilde{X}) \rightarrow \pi(\tilde{X})$$

such that $\pi(p) \circ \tilde{\star} = \pi(\star) \circ (\pi p \times \pi p)$, where \star 's are the 2-ary operations of X . By [4, 10.5.5] the operations $\tilde{\star}$ induce continuous 2-ary operations on \tilde{X} . Similarly, there are 1-ary operations

$$\tilde{\omega}: \pi(\tilde{X}) \rightarrow \pi(\tilde{X})$$

such that $(\pi p) \circ \tilde{\omega} = (\pi \omega) \circ \pi p$, where ω 's are the 1-ary operations of X and the operations $\tilde{\omega}$ induce continuous 1-ary operations on \tilde{X} . So \tilde{X} becomes a group with operations and by the lifted topology the operations are continuous. Hence \tilde{X} becomes a topological group with operations.

Since the category Cov_{TC}/X of covers of X in the category of topological groups is equivalent to the category $\text{Cov}_{\text{GrGd}}/\pi(X)$ of covers of $\pi(X)$ in the category of group-groupoids, by the following diagram the proof is completed

$$\begin{array}{ccc} \text{Cov}_{\text{TC}}/X & \xrightarrow{\pi} & \text{Cov}_{\text{Cat}(\mathbf{C})}/\pi(X) \\ \downarrow & & \downarrow \\ \text{Cov}_{\text{TGr}}/X & \xrightarrow{\pi} & \text{Cov}_{\text{GrGd}}/\pi(X). \end{array}$$

The proof of theorem is complete. □

Let G be an internal groupoid in \mathbf{C} . Let $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ be the category of covers of G in the category $\text{Cat}(\mathbf{C})$ of internal categories in \mathbf{C} . So the objects of $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ are the covering morphisms $p: \tilde{G} \rightarrow G$ over G of internal groupoids and a morphism from $p: \tilde{G} \rightarrow G$ to $q: \tilde{H} \rightarrow G$ is a morphism $f: \tilde{G} \rightarrow \tilde{H}$ of internal groupoids, which becomes also a covering morphism, such that $qf = p$.

Let $\text{Act}_{\text{Cat}(\mathbf{C})}/G$ be the category of internal groupoid actions of G on groups with operations. So, an object of $\text{Act}_{\text{Cat}(\mathbf{C})}/G$ is an internal groupoid action (X, θ, φ) of G and a morphism from (X, θ, φ) to (Y, θ', φ') is a morphism $f: X \rightarrow Y$ of groups with operations such that $\theta = \theta'f$ and $f(xa) = (fx)a$ whenever xa is defined, for any objects (X, θ, φ) and (Y, θ', φ') of $\text{Act}_{\text{Cat}(\mathbf{C})}/G$.

Theorem 4.2. *For an internal groupoid G in \mathbf{C} , the categories $\text{Act}_{\text{Cat}(\mathbf{C})}/G$ and $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ are equivalent.*

Proof. By Theorem 3.4 for an internal groupoid action (X, θ, φ) of G , we have a morphism $p: G \ltimes X \rightarrow G$ of internal groupoids, which is a covering morphism on underlying groupoids. This gives a functor

$$\Gamma: \text{Act}_{\text{Cat}(\mathbf{C})}/G \rightarrow \text{Cov}_{\text{Cat}(\mathbf{C})}/G.$$

Conversely, if $p: \tilde{G} \rightarrow G$ is a covering morphism of internal groupoids, then by Example

3.4 we have an internal groupoid action. In this way we define a functor

$$\Phi: \text{Cov}_{\text{Cat}(\mathbf{C})}/G \rightarrow \text{Act}_{\text{Cat}(\mathbf{C})}/G.$$

The natural equivalences $\Gamma\Phi \simeq 1$ and $\Phi\Gamma \simeq 1$ follow. \square

5 Covers of crossed modules in groups with operations

The conditions of a crossed module in groups with operations are formulated in [25, Proposition 2] as follows.

Definition 5.1. A *crossed module* in \mathbf{C} is $\alpha: A \rightarrow B$ is a morphism in \mathbf{C} , where B acts on A (i.e. we have a derived action in \mathbf{C}) with the conditions for any $b \in B$, $a, a' \in A$, and $\star \in \Omega'_2$:

$$\text{CM1} \quad \alpha(b \cdot a) = b + \alpha(a) - b;$$

$$\text{CM2} \quad \alpha(a) \cdot a' = a + a' - a;$$

$$\text{CM3} \quad \alpha(a) \star a' = a \star a';$$

$$\text{CM4} \quad \alpha(b \star a) = b \star \alpha(a) \text{ and } \alpha(a \star b) = \alpha(a) \star b.$$

A *morphism* from $\alpha: A \rightarrow B$ to $\alpha': A' \rightarrow B'$ is a pair $f_1: A \rightarrow A'$ and $f_2: B \rightarrow B'$ of morphisms in \mathbf{C} such that

1. $f_2\alpha(a) = \alpha'f_1(a),$
2. $f_1(b \cdot a) = f_2(x) \cdot f_1(a),$
3. $f_1(b \star a) = f_2(x) \star f_1(a)$

for any $x \in B$, $a \in A$ and $\star \in \Omega'_2$. So the category XMod of crossed modules in groups with operation is obtained.

The following theorem was proved in [25, Theorem1].

Theorem 5.1. *The category XMod of crossed modules and the category $\text{Cat}(\mathbf{C})$ of internal groupoids in \mathbf{C} are equivalent.*

By Theorem 5.1, evaluating the covering morphism of internal groupoids (Definition 3.3) in terms of the corresponding morphism of crossed modules in \mathbf{C} , we can obtain the notion of a covering morphism of crossed modules in \mathbf{C} . If $f: H \rightarrow G$ is a covering morphism of internal groupoids in \mathbf{C} as defined in Definition 3.3 and (f_1, f_2) is the morphism of crossed modules corresponding to f , then $f_1: A \rightarrow A'$ is an isomorphism in \mathbf{C} , where $A = \text{St}_H 0$, $A' = \text{St}_G 0$ and f_1 is the restriction of f . Therefore we call a morphism (f_1, f_2) of crossed modules from $\alpha: A \rightarrow B$ to $\alpha': A' \rightarrow B'$ in \mathbf{C} as a *cover* if $f_1: A \rightarrow A'$ is an isomorphism in \mathbf{C} .

Theorem 5.2. *Let G be an internal groupoid in \mathbf{C} and $\alpha: A \rightarrow B$ the crossed module in \mathbf{C} corresponding to G . Let $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ be the category of covers of G in the category $\text{Cat}(\mathbf{C})$ of internal groupoids in \mathbf{C} and let $\text{Cov}_{\text{XMod}}/(\alpha: A \rightarrow B)$ be the category of covers of $\alpha: A \rightarrow B$ in \mathbf{C} . Then the categories $\text{Cov}_{\text{Cat}(\mathbf{C})}/G$ and $\text{Cov}_{\text{XMod}}/(\alpha: A \rightarrow B)$ are equivalent.*

Proof. The proof is obtained from Theorem 5.1 and therefore the details are omitted. \square

Let X be an object of \mathbf{TC} . Then by Example 3.2, $\pi(X)$ is an internal groupoid and therefore $d_1: St_{\pi(X)}0 \rightarrow X$ is a crossed module in \mathbf{C} . So as a result of Theorems 4.1 and 5.2 we can obtain the following corollary.

Corollary 5.1. *If X is a topological group with operations in \mathbf{TC} whose underlying topology has a simply connected cover, then the category $\text{Cov}_{\mathbf{TC}}/X$ of covers of X in the category \mathbf{TC} of topological groups with operations and the category $\text{Cov}_{\text{XMod}}/(d_1: St_{\pi(X)}0 \rightarrow X)$ of covers of the crossed module $d_1: St_{\pi(X)}0 \rightarrow X$ in \mathbf{C} are equivalent.*

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